AIAA 80-1659R

A Short Efficient Analytical Satellite Theory

F.R. Hoots*
Aerospace Defense Command, Colorado Springs, Colo.

A simplified analytical solution for the motion of an artificial Earth satellite under the combined influences of the zonal harmonics through J_4 and a nonrotating spherically symmetric power density function atmosphere is given. This solution has been obtained by simplification of a more extensive analytical solution previously published by the author. The simplified version has nearly the same accuracy, yet has only about 16% the formula amount of the full theory. Comparison is made with two other widely used simplfied analytical theories of comparable size. Our solution is found to consistently produce equal or superior prediction accuracies. For low-altitude, high-drag satellites, the improvement can approach a factor of 10 or more.

Introduction

IN many applications where satellite ephemerides are generated on a routine large-scale basis it is desired to have a general perturbations model of moderate accuracy, yet one which has a fast computer runtime and small computer storage requirements. One of the first theories of this type was developed by Hilton and Kuhlman[1] in 1966. (For a more recent reference see Hoots and Roehrich.[2]) This simplified general perturbations theory, hereafter called SGP, uses a simplification of the work of Kozai[3] for its gravitational model and it takes the drag effect on mean motion as linear in time. This assumption dictates a quadratic variation of mean anomaly with time. The drag effect on eccentricity is modeled in such a way that perigee height remains constant. SGP is used throughout the world at radar installations, data collection stations, universities, etc., where fast, moderate accuracy satellite predictions are required.

In 1970 Cranford (see Lane and Hoots[4]) developed a simplified general perturbations theory called SGP4. This model was obtained by simplification of the more extensive analytical theory of Lane and Cranford,[5] which uses the solution of Brouwer[6] for its gravitational model and a power density function for its atmospheric model. SGP4 is currently used by NORAD (North American Aerospace Defense Command) for updating and maintenance of the entire inventory of near-Earth satellites. Although there are other general perturbations theories, these two are the main ones which, owing to their application, have been designed for very fast computer runtime and a computer core size under 2K words.

Although SGP and SGP4 can generate predictions of moderate accuracy for most satellites, it has been found that their predictions degrade rapidly as satellites approach decay. In light of this, we have developed a general perturbations model, hereafter called SGP8, which has comparable computer runtime and storage requirements, yet provides significantly improved prediction accuracy near decay.

Simplification Procedure

The SGP8 theory is obtained by simplification of an extensive analytical theory of Hoots[7] which uses the same gravitational and atmospheric models as Lane and Cranford did but integrates the differential equations in a much different manner. The full theory is valid for all eccentricities between 0 and 0.1 and all inclinations not near 0 deg or critical inclination.† In obtaining the simplified set of equations, the guiding factor was that SGP8 should have a computer runtime and core size comparable to SGP and SGP4, yet should retain most of the prediction accuracy of the full theory. Since the full theory contains several terms which only become important for larger eccentricities, it was felt many terms could be dropped without affecting predictions on most satellites. Furthermore, it can be shown that many terms included in the full theory are much smaller than the differences introduced by using the power density function atmosphere to model the real-world atmosphere. Thus, in an operational environment, many of the differences between the full theory and the simplified theory will be masked by differences between the model and the real world.

In order to determine the simplified set of equations, a set of reference orbits was generated using an eighth-order Cowell numerical integration with a 1-min step size. The equations of motion had the power density function for the atmospheric model and the zonal harmonics through J_4 for the gravitational model. The reference orbit parameters were selected to span the range of orbital elements for which the full theory was originally developed.

The standard for comparison was established by testing the performance of the full model against the reference orbits. Since an exact transformation from osculating to mean elements is not available, a set of seven constants was determined through a least-squares fit of the analytical theory to the first day of the given reference orbit. The constants are initial values of the "mean" mean motion, eccentricity, inclination, mean anomaly, argument of perigee, longitude of ascending node, and the drag parameter $B^* = \frac{1}{2}\rho_0 C_D A/m$, respectively, where C_D is a dimensionless drag coefficient, A is the effective cross-sectional area of the satellite of mass m, and ρ_0 is a reference density value for the power density function. These constants were then used to make a prediction with the analytical theory over the second day of the given reference orbit. The vector magnitude differences between the analytical theory and the numerical reference for both the fit and prediction spans were then computed and plotted.

As terms to be dropped from the full theory were identified, numerical comparisons were made with the reference orbits in the same manner as described above. Only a term whose absence gave minimal degradation throughout the range of reference orbits was confirmed numerically as a term to be excluded from the full theory. In this manner we arrived at a simplified theory with fast runtime and small size and only a slight loss of accuracy for the more eccentric satellites.

Before discussing the terms to be dropped from the full theory, let us outline the solution of Hoots. This solution used a portion of the work of Brouwer[6] as modified by Lyddane[8] along with application of the method of averaging to provide a sequence of transformations to a set of triple-primed variables whose differential equations contain only secular and long-period variations through second order. All short-period variations were removed by the transformations and

Presented as Paper 80-1659 at the AIAA/AAS Astrodynamics Conference, Danvers, Mass., Aug. 11-13, 1980; submitted Sept. 11, 1980; revision received Sept. 21, 1981.

*Astrodynamicist, Directorate of Astrodynamics. Member AIAA.

†Approximately 90% of the satellites currently in orbit satisfy all these restrictions.

are given explicitly through first order by the transformation equations. Here we have taken J_2 and drag to be first order and the other zonal harmonics to be second order. The triple-primed differential equations were then integrated analytically and this solution in conjunction with the transformations gives equations for computing the six osculating orbital elements which describe the orbital motion of the satellite.

Predictions are made by first calculating the secular effects of gravity and atmospheric drag through the equations

$$n''' = f_1(t) \qquad e''' = f_2(t) \qquad I''' = f_3(t)$$
$$\omega''' = f_4(t) \qquad \Omega''' = f_5(t) \qquad M''' = f_6(t) \qquad (1)$$

where the functions f_i are explicit functions of the time t and the constants n_0''', e_0''', I_0''', ω_0''', Ω_0''', M_0''', which are the values at the time t_0 of the triple-primed mean motion, eccentricity, inclination, argument of perigee, longitude of ascending node, and mean anomaly, respectively. Here we have introduced the convention that a subscript 0 denotes the value at time t_0.

A change of variables is then made to the alternate set

$$x_1''' = e''' \sin\omega''' \quad x_2''' = e''' \cos\omega''' \quad y_1''' = M''' + \omega''' \qquad (2)$$

and the transformation to double-primed variables is made by

$$n'' = n''' + \delta n_D \qquad x_1'' = x_1''' + \delta x_{1D} \qquad x_2'' = x_2''' + \delta x_{2D}$$
$$I'' = I''' \qquad \Omega'' = \Omega''' \qquad y_1'' = y_1''' + \delta y_{1D} \qquad (3)$$

where δn_D, δx_{1D}, δx_{2D}, δy_{1D} are first-order periodic variations due to drag and are functions of the triple-primed mean variables. Next, a change of variables is made to a set which is needed for the Lyddane-modified Brouwer transformation. These variables are

$$z_1'' = (\mu/n''^2)^{1/3} \qquad z_2'' = e'' \sin M'' \qquad z_3'' = e'' \cos M''$$
$$z_4'' = \sin I''/2 \, \sin\Omega'' \qquad z_5'' = \sin I''/2 \, \cos\Omega'' \qquad z_6'' = y_1'' + \Omega'' \qquad (4)$$

where μ is the product of Newton's gravitational constant and the mass of the Earth

$$e'' = \sqrt{x_1''^2 + x_2''^2} \qquad \omega'' = \tan^{-1}(x_1''/x_2'') \qquad M'' = y_1'' - \omega'' \quad (5)$$

The transformation to the osculating state is then given by

$$z_1 = z_1'' + \delta z_1 \qquad z_2 = z_2'' + \delta z_2 \qquad z_3 = z_3'' + \delta z_3$$
$$z_4 = z_4'' + \delta z_4 \qquad z_5 = z_5'' + \delta z_5 \qquad z_6 = z_6'' + \delta z_6 \qquad (6)$$

where the δz_i are first-order periodic variations due to geopotential and are functions of the double-primed variables. Osculating orbital elements can then be calculated by

$$e = \sqrt{z_2^2 + z_3^2} \qquad \Omega = \tan^{-1}(z_4/z_5) \qquad \omega = z_6 - M - \Omega \quad (7)$$
$$M = \tan^{-1}(z_2/z_3) \qquad I = 2\sin^{-1}(\sqrt{z_4^2 + z_5^2}) \qquad a = z_1$$

where a, e, I, M, ω, Ω are the osculating semimajor axis, eccentricity, inclination, mean anomaly, argument of perigee, and longitude of ascending node, respectively. Osculating position and velocity can then be calculated in the usual manner.

The first type of term to be considered for simplification was the short-period drag periodics of Eqs. (3). It was found that these terms could be excluded from the theory with negligible effect. Second, we dropped terms of the form of drag coupled with drag which appear in the secular Eqs. (1) for mean anomaly and argument of perigee. The next type of term examined was of the form of drag coupled with gravity. Although terms of this form occur in the triple-primed differential equations for all six orbital elements, it was found that the direct effect on each of the elements was very small. However, since changes in the mean motion and eccentricity directly cause changes in total drag and since the mean motion is integrated a second time in the mean anomaly equation, it was found that the dominant part of the coupled terms must be retained in the mean motion and eccentricity differential equations, but all coupled terms can be neglected in the other four differential equations.

It is well known that the gravitational model of Vinti[9] allows algebraic combination of the second and fourth zonal harmonic terms. Since the Vinti potential differs from the true gravitational coefficients, we found that adopting the Vinti relationship $J_4 = -J_2^2$ caused measurable degradation. However, if we use the Vinti potential in the long-period terms only and retain the values of J_2 and J_4 in the secular terms, we found a significant algebraic simplification in our equations with little loss of accuracy. In addition, adopting the Vinti potential removes from SGP8 any singularities at the critical inclination.

The final simplification concerned the purely gravitational terms. It was found that terms of size second order times e^2 could be neglected in the secular gravitational terms. Additionally, the gravitational periodics can be simplified considerably. It has been shown by Hoots[10] that the Lyddane-modified Brouwer geopotential transformation can be reformulated in terms of an alternate set of variables which allows a direct conversion from double-primed elements to Cartesian position and velocity while reducing the formula amount of the transformation by one-third. By using this alternate set of variables and retaining only the dominant periodic terms, we obtained a significant decrease in the number of terms in the transformation while sacrificing little in accuracy for most satellites.

Simplified Equations

The final set of equations which incorporates all the simplifications discussed in the previous section is summarized. Given initial values of the triple-primed orbital elements and the drag parameter, predictions at time t are made by first calculating the constants

$$p = \frac{2\ddot{n}_0^2 - \dot{n}_0 \dddot{n}_0}{\ddot{n}_0^2 - \dot{n}_0 \dddot{n}_0} \qquad \gamma = -\frac{\ddot{n}_0}{\dot{n}_0} \frac{1}{(p-2)}$$

$$n_D = n_0 + \dot{n}_0/p\gamma \qquad q = 1 - \ddot{e}_0/\dot{e}_0\gamma \qquad e_D = e_0 + \dot{e}_0/q\gamma$$

$$\dot{M}_1 = -\frac{3}{2}\frac{nk_2}{a^2\beta^3}(1 - 3\theta^2)$$

$$\dot{M}_2 = \frac{3}{16}\frac{nk_2^2}{a^4\beta^7}(13 - 78\theta^2 + 137\theta^4)$$

$$\dot{\omega}_1 = -\frac{3}{2}\frac{nk_2}{a^2\beta^4}(1 - 5\theta^2)$$

$$\dot{\omega}_2 = \frac{3}{16}\frac{nk_2^2}{a^4\beta^8}(7 - 114\theta^2 + 395\theta^4) + \frac{5}{4}\frac{nk_4}{a^4\beta^8}(3 - 36\theta^2 + 49\theta^4)$$

$$\dot{\Omega}_1 = -3\frac{nk_2}{a^2\beta^4}\theta$$

$$\dot{\Omega}_2 = \frac{3}{2}\frac{nk_2^2}{a^4\beta^8}\theta(4 - 19\theta^2) + \frac{5}{2}\frac{nk_4}{a^4\beta^8}\theta(3 - 7\theta^2) \qquad (8)$$

where all quantities are triple-primed variables evaluated at t_0, where $\dot{n}_0$, $\ddot{n}_0$, $\dddot{n}_0$, $\dot{e}_0$, $\ddot{e}_0$ are given in the Appendix, where

$$\theta = \cos I_0''' \qquad k_2 = \tfrac{1}{2} J_2 R^2 \qquad k_4 = -\tfrac{3}{8} J_4 R^4$$

and where R is the Earth equatorial radius.

Triple-primed elements at time t can be calculated by

$$n''' = n_D - (n_D - n_0''')[1 - \gamma(t - t_0)]^p$$

$$e''' = e_D + (e_0''' - e_D)[1 - \gamma(t - t_0)]^q$$

$$I''' = I_0'''$$

$$\omega''' = \omega_0''' + \dot{\omega}_1 \left[(t - t_0) + \frac{7}{3} \frac{1}{n_0'''} Z_1 \right] + \dot{\omega}_2 (t - t_0)$$

$$\Omega''' = \Omega_0''' + \dot{\Omega}_1 \left[(t - t_0) + \frac{7}{3} \frac{1}{n_0'''} Z_1 \right] + \dot{\Omega}_2 (t - t_0)$$

$$M''' = M_0''' + n_0'''(t - t_0) + Z_1 + \dot{M}_1 \left[(t - t_0) + \frac{7}{3} \frac{1}{n_0'''} Z_1 \right]$$
$$+ \dot{M}_2 (t - t_0) \qquad (9)$$

where

$$Z_1 = \frac{\dot{n}_0}{p\gamma} \left((t - t_0) + \frac{1}{\gamma(p+1)} \{ [1 - \gamma(t - t_0)]^{p+1} - 1 \} \right)$$

The geopotential periodics are included by first calculating the quantities

$$r''' = \frac{a''' \beta'''^2}{1 + e''' \cos f'''} \qquad \dot{r}''' = \frac{n''' a''' e'''}{\beta'''} \sin f'''$$

$$(r\dot{f})''' = \frac{n''' a'''^2 \beta'''}{r'''} \qquad \lambda''' = u''' + \Omega'''$$

$$y_4''' = \sin I'''/2 \quad \sin u'''$$

$$y_5''' = \sin I'''/2 \quad \cos u''' \qquad (10)$$

where

$$a''' = (\mu / n'''^2)^{1/3} \qquad \beta''' = (1 - e'''^2)^{1/2} \qquad u''' = f''' + \omega'''$$

and where f''' is the mean true anomaly at time t and has the same functional relationship to e''' and M''' as the osculating true anomaly f has to e and M. The transformation to the osculating state is given by

$$r = r''' + \delta r \qquad \dot{r} = \dot{r}''' + \delta \dot{r}$$

$$r\dot{f} = (r\dot{f})''' + \delta(r\dot{f}) \qquad \lambda = \lambda''' + \delta\lambda$$

$$y_4 = y_4''' + \cos u''' \sin I'''/2 \quad \delta u + \tfrac{1}{2} \sin u''' \cos I'''/2 \quad \delta I$$

$$y_5 = y_5''' - \sin u''' \sin I'''/2 \quad \delta u + \tfrac{1}{2} \cos u''' \cos I'''/2 \quad \delta I \quad (11)$$

where

$$\delta r = \frac{1}{2} \frac{k_2}{a\beta^2} [(1 - \theta^2)\cos 2u + 3(1 - 3\theta^2)] - \frac{1}{4} \frac{A_{3,0}}{k_2} \sin I \sin u$$

$$\delta \dot{r} = -n \left(\frac{a}{r} \right)^2 \left[\frac{k_2}{a\beta^2} (1 - \theta^2) \sin 2u + \frac{1}{4} \frac{A_{3,0}}{k_2} \sin I \cos u \right]$$

$$\delta I = \theta \left[\frac{3}{2} \frac{k_2}{a^2 \beta^4} \sin I \cos 2u - \frac{1}{4} \frac{A_{3,0}}{k_2 a \beta^2} e \sin \omega \right]$$

$$\delta(r\dot{f}) = -n \left(\frac{a}{r} \right)^2 \delta r + na \left(\frac{a}{r} \right) \frac{\sin I}{\theta} \delta I$$

$$\delta u = \frac{1}{2} \frac{k_2}{a^2 \beta^4} [\tfrac{1}{2}(1 - 7\theta^2)\sin 2u - 3(1 - 5\theta^2)(f - M + e\sin f)]$$
$$- \frac{1}{4} \frac{A_{3,0}}{k_2 a \beta^2} \left[\sin I \cos u (2 + e\cos f) + \frac{1}{2} \frac{\theta^2}{\sin I/2 \cos I/2} e \cos \omega \right]$$

$$\delta\lambda = \frac{1}{2} \frac{k_2}{a^2 \beta^4} [\tfrac{1}{2}(1 + 6\theta - 7\theta^2)\sin 2u$$
$$- 3(1 + 2\theta - 5\theta^2)(f - M + e\sin f)]$$
$$+ \frac{1}{4} \frac{A_{3,0}}{k_2 a \beta^2} \sin I \left[\frac{e\theta}{1 + \theta} \cos \omega - (2 + e\cos f)\cos u \right] \qquad (12)$$

in which all quantities are triple-primed elements at time t and $A_{3,0} = -J_3 R^3$.

Cartesian position and velocity can now be calculated directly. Let U be a unit vector in the direction of the osculating radius vector, and let V be a unit vector in the osculating orbital plane such that U, V, and the osculating angular momentum form a right-handed orthogonal system. Then

$$U_x = 2y_4(y_5 \sin\lambda - y_4 \cos\lambda) + \cos\lambda$$

$$U_y = -2y_4(y_5 \cos\lambda + y_4 \sin\lambda) + \sin\lambda$$

$$U_z = 2y_4 \cos I/2$$

$$V_x = 2y_5(y_5 \sin\lambda - y_4 \cos\lambda) - \sin\lambda$$

$$V_y = -2y_5(y_5 \cos\lambda + y_4 \sin\lambda) + \cos\lambda$$

$$V_z = 2y_5 \cos I/2 \qquad (13)$$

where $U_x, U_y, U_z, V_x, V_y, V_z$ denote the x,y,z components of the vectors U and V, respectively, and where

$$\cos I/2 = \sqrt{1 - y_4^2 - y_5^2}$$

The osculating position r and velocity $\dot{r}$ are given by

$$r = rU \qquad \dot{r} = \dot{r}U + r\dot{f}V \qquad (14)$$

Comparison with Other Theories

The SGP8 equations have retained most of the prediction accuracy of the full theory except for satellites with larger eccentricity. In particular for satellites with $e < 0.05$, the average prediction degradation for the simplified theory is less than 2%. Additionally, the computer runtime and core size of SGP8 is comparable to SGP and SGP4.

The performance of the three simplified theories (SGP8, SGP4, SGP) cannot be determined by comparison with the reference orbits discussed above, since all three theories were derived from slightly different sets of equations of motion. Moreover, since ultimately a satellite theory will be used in a real-world environment, it was decided to use a set of simulated real-world test cases to compare the three simplified theories.

The simulated real-world reference orbits (inertial position at 10-min intervals) were generated using the same numerical integrator discussed in the second section and had for a gravitational model a twentieth-order truncation of the Goddard Earth Model 9 (see Lerch et al.[11]). The atmospheric model used was the Jacchia[12] model with constant solar flux of $F_{10.7} = 150$ (units of 10^{-22} W/m^2/cycle/s) and constant geomagnetic index of $a_p = 0$. All reference orbits were generated with a drag coefficient $C_D A/m = 0.02$ m^2/kg, an

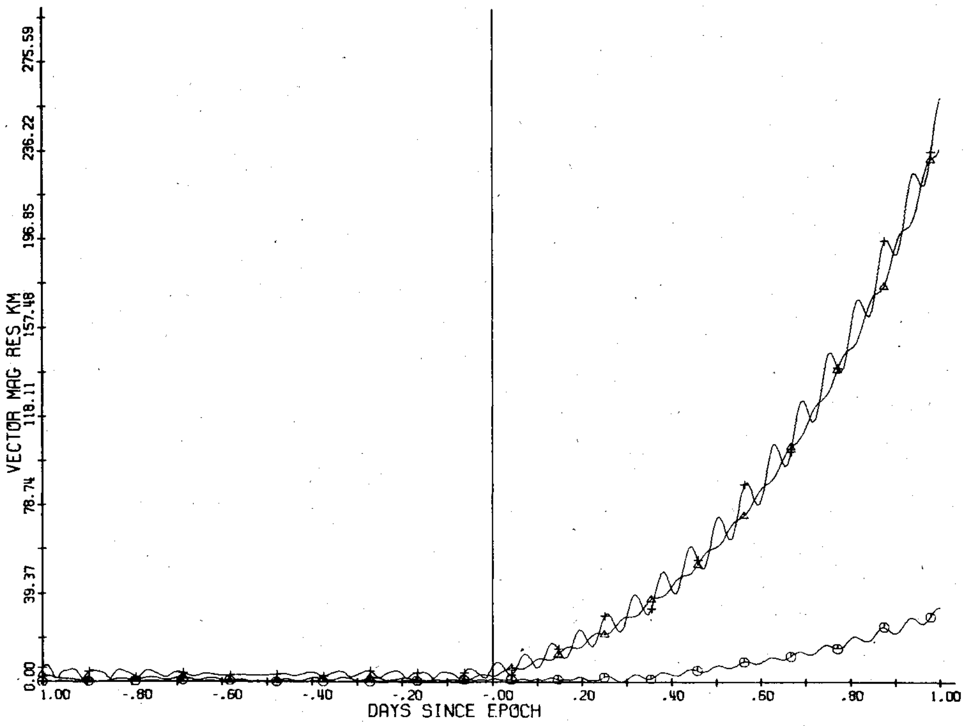

Fig. 1 Vector magnitude residual vs time for case no. 7.

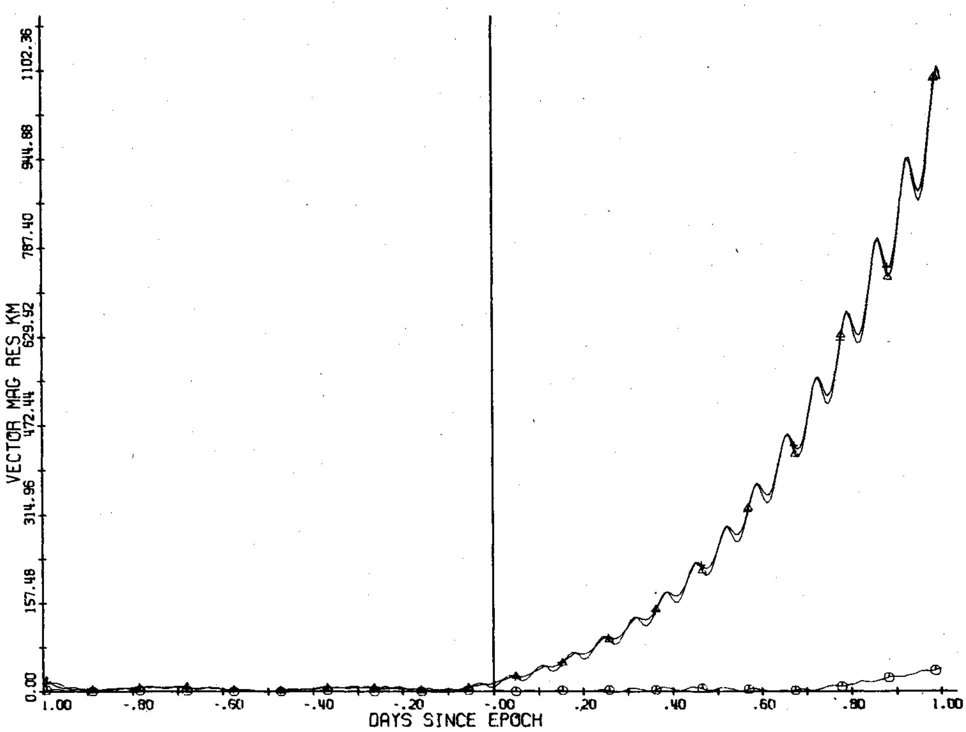

Fig. 2 Vector magnitude residual vs time for case no. 14.

initial longitude of ascending node of 0 deg, and an initial argument of perigee of 30 deg.

Five eccentricity bands were selected to span the range between 0 and 0.1, with three different inclinations for each eccentricity band. Finally, semimajor axis was selected so that the first point of each reference orbit would be four days before decay. These orbital characteristics are summarized in Table 1. The three models (SGP8, SGP4, SGP) were compared with the reference orbits by determining for each model a set of seven constants (six orbital elements and a drag parameter) through a least-squares fit of the theory to the first day of the given reference orbit. These constants were then used to make a prediction over the second day of the reference orbit. In Table 2 the rms of the vector magnitude differences between each theory and each reference orbit is tabulated for both the fit span and the prediction span. The fit rms is given first, followed by the prediction rms, with all in units of kilometers. The epoch time for each case is at the end of the fit span and is three days before decay. For illustration, graphs of vector magnitude differences for test cases 7 (Fig. 1) and 14 (Fig. 2) are given. The symbology is an octagon for SGP8, a triangle for SGP4, and a plus for SGP.

Table 1 Orbital characteristics

Case no.	Epoch perigee, km	Epoch apogee, km	Epoch eccentricity	Epoch inclination, deg
1	235	243	0.0007	15
2	230	242	0.0009	45
3	218	237	0.0014	75
4	222	259	0.0028	15
5	209	256	0.0036	45
6	195	253	0.0044	75
7	203	294	0.0069	15
8	191	295	0.0079	45
9	178	298	0.0090	75
10	154	718	0.0414	15
11	142	693	0.0405	45
12	132	809	0.0495	75
13	140	1334	0.0839	15
14	129	1241	0.0788	45
15	121	1315	0.0842	75

Table 2 Fit/prediction rms

Case no.	SGP8	SGP4	SGP
1	0.85/2.21	1.64/6.25	6.93/104.04
2	0.68/3.71	0.88/3.80	4.52/103.80
3	1.07/9.27	1.43/97.94	4.15/99.86
4	0.63/4.24	0.75/5.16	4.41/101.11
5	0.70/17.36	1.50/98.29	3.37/102.69
6	0.96/21.30	1.54/97.57	3.04/105.23
7	0.61/13.21	1.44/106.97	3.92/111.97
8	0.69/33.44	1.45/102.90	2.49/108.10
9	0.87/25.57	1.56/110.34	2.54/109.72
10	0.73/11.34	1.93/127.31	3.13/129.99
11	0.87/53.50	3.14/274.25	3.45/275.79
12	1.15/33.28	4.83/347.94	4.60/345.60
13	2.12/67.42	4.13/189.07	4.54/190.80
14	1.76/14.63	7.02/463.43	7.11/465.65
15	2.31/54.04	7.53/672.31	6.53/667.49

Conclusions

A simplified general perturbations theory has been developed which is fast, efficient, requires minimal computer storage space, and provides low altitude satellite prediction accuracies which are significantly improved over other widely used general perturbations theories. For satellites within a few days of decay, the average prediction accuracy improvement is a factor of 11 over other analytical theories. The new equations have maintained much of the prediction accuracy of the full theory, yet have only about 16% the total formula amount, resulting in a computer runtime and core size nearly the same as other simplified theories.

The simulation of the real-world has purposely avoided such variables as unmodeled density variations, data noise and biases, and data gaps, since these would tend to cloud the issue of which theory is the better mathematical model. When these are included, the enhancements may not appear as dramatic in every case but should still consistently provide an improvement over other general perturbations theories.

Acknowledgments

I am especially indebted to Linda Crawford for her invaluable and tenacious assistance in the generation of and comparisons with the reference orbits. The software for this effort was provided by Don Larson and Richard France, to whom I express my sincere thanks.

Appendix

The equations for calculating the constants $\dot{n}_0'''$, $\ddot{n}_0'''$, $\dddot{n}_0'''$, $\dot{e}_0'''$, and $\ddot{e}_0'''$ are given below. All quantities are triple-primed variables evaluated at t_0. The parameters q_0 and s are adjustable parameters for the power density function.

$$B = B^*(q_0 - s)^4 \qquad \beta^2 = 1 - e^2 \qquad a = (\mu/n^2)^{1/3} \qquad \xi = 1/(a\beta^2 - s) \qquad \eta = es\xi \qquad \psi = \sqrt{1-\eta^2} \qquad \theta = \cos I$$

$$\alpha^2 = 1 + e^2 \qquad C_0 = Bna\xi^4\alpha^{-1}\psi^{-7} \qquad C_1 = (3/2)n\alpha^4 C_0 \qquad D_1 = \xi\psi^{-2}/a\beta^2 \qquad D_2 = 12 + 36\eta^2 + (9/2)\eta^4$$

$$D_3 = 15\eta^2 + (5/2)\eta^4 \qquad D_4 = 5\eta + (15/4)\eta^3 \qquad D_5 = \xi\psi^{-2} \qquad B_1 = -k_2(1 - 3\theta^2) \qquad B_2 = -k_2(1 - \theta^2)$$

$$B_3 = (A_{3,0}/k_2)\sin I \qquad C_2 = D_1 D_3 B_2 \qquad C_3 = D_4 D_5 B_3 \qquad C_4 = D_1 D_7 B_2 \qquad C_5 = D_5 D_8 B_3$$

$$\dot{n}_0 = C_1(2 + 3\eta^2 + 20e\eta + 5e\eta^3 + (17/2)e^2 + 34e^2\eta^2 + D_1 D_2 B_1 + C_2\cos 2\omega + C_3 \sin\omega)$$

$$D_6 = 30\eta + (45/2)\eta^3 \qquad D_7 = 5\eta + (25/2)\eta^3 \qquad D_8 = 1 + (27/4)\eta^2 + \eta^4$$

$$\dot{e}_0 = -C_0(4\eta + \eta^3 + 5e + 15e\eta^2 + (31/2)e^2\eta + 7e^2\eta^3 + D_1 D_6 B_1 + C_4 \cos 2\omega + C_5 \sin\omega) \qquad \dot{\alpha}/\alpha = e\dot{e}\alpha^{-2} \qquad C_6 = \tfrac{1}{3}\dot{n}/n$$

$$\dot{\xi}/\xi = 2a\xi(C_6\beta^2 + e\dot{e}) \qquad \dot{\eta} = (\dot{e} + e\dot{\xi}/\xi)s\xi \qquad \dot{\psi}/\psi = -\eta\dot{\eta}\psi^{-2} \qquad \dot{C}_0/C_0 = C_6 + 4\dot{\xi}/\xi - \dot{\alpha}/\alpha - 7\dot{\psi}/\psi$$

$$\dot{C}_1/C_1 = \dot{n}/n + 4\dot{\alpha}/\alpha + \dot{C}_0/C_0 \qquad D_9 = 6\eta + 20e + 15e\eta^2 + 68e^2\eta \qquad D_{10} = 20\eta + 5\eta^3 + 17e + 68e\eta^2 \qquad D_{11} = 72\eta + 18\eta^3$$

$$D_{12} = 30\eta + 10\eta^3 \qquad D_{13} = 5 + (45/4)\eta^2 \qquad D_{14} = \dot{\xi}/\xi - 2\dot{\psi}/\psi \qquad D_{15} = 2(C_6 + e\dot{e}\beta^{-2})$$

$$\dot{D}_1 = D_1(D_{14} + D_{15}) \qquad \dot{D}_2 = \dot{\eta}D_{11} \qquad \dot{D}_3 = \dot{\eta}D_{12} \qquad \dot{D}_4 = \dot{\eta}D_{13} \qquad \dot{D}_5 = D_5 D_{14}$$

$$\dot{C}_2 = B_2(\dot{D}_1 D_3 + D_1\dot{D}_3) \qquad \dot{C}_3 = B_3(\dot{D}_5 D_4 + D_5\dot{D}_4) \qquad \dot{\omega} = -(3/2)(nk_2/a^2\beta^4)(1 - 5\theta^2) \qquad \ddot{n}_0 = \dot{n}\dot{C}_1/C_1 + C_1 D_{16}$$

$$D_{16} = D_9\dot{\eta} + D_{10}\dot{e} + B_1(\dot{D}_1 D_2 + D_1\dot{D}_2) + \dot{C}_2\cos 2\omega + \dot{C}_3\sin\omega + \dot{\omega}(C_3\cos\omega - 2C_2\sin 2\omega)$$

$$\ddot{e}_0 = \dot{e}\dot{C}_0/C_0 - C_0\Big([4 + 3\eta^2 + 30e\eta + (31/2)e^2 + 21e^2\eta^2]\dot{\eta} + (5 + 15\eta^2 + 31e\eta + 14e\eta^3)\dot{e}$$
$$+ B_1\{\dot{D}_1 D_6 + D_1\dot{\eta}[30 + (135/2)\eta^2]\} + B_2\{\dot{D}_1 D_7 + D_1\dot{\eta}[5 + (75/2)\eta^2]\}\cos 2\omega$$
$$+ B_3[\dot{D}_5 D_8 + D_5\eta\dot{\eta}(27/2 + 4\eta^2)]\sin\omega + \dot{\omega}(C_5\cos\omega - 2C_4\sin 2\omega)\Big)$$

$$D_{17}=\ddot{n}/n-(\dot{n}/n)^2 \qquad \ddot{\xi}/\xi=2(\dot{\xi}/\xi-C_6)\dot{\xi}/\xi+2a\xi(\tfrac{1}{3}D_{17}\beta^2-2C_6e\dot{e}+\dot{e}^2+e\ddot{e}) \qquad \ddot{\eta}=(\ddot{e}+2\dot{e}\dot{\xi}/\xi)s\xi+\eta\ddot{\xi}/\xi$$

$$D_{18}=\ddot{\xi}/\xi-(\dot{\xi}/\xi)^2 \qquad D_{19}=-(\dot{\psi}/\psi)^2(1+\eta^{-2})-\eta\ddot{\eta}\psi^{-2}$$

$$\ddot{D}_1=\dot{D}_1(D_{14}+D_{15})+D_1(D_{18}-2D_{19})+\tfrac{2}{3}D_{17}+2\alpha^2\dot{e}^2\beta^{-4}+2e\ddot{e}\beta^{-2})$$

$$\begin{aligned}\ddot{n}_0=\dot{n}[(4/3)D_{17}+3\dot{e}^2\alpha^{-2}+3e\ddot{e}\alpha^{-2}-6(\dot{\alpha}/\alpha)^2+4D_{18}-7D_{19}]+\ddot{n}\dot{C}_1/C_1\\ +C_1\big(D_{16}\dot{C}_1/C_1+D_9\ddot{\eta}+D_{10}\ddot{e}+\dot{\eta}^2(6+30e\eta+68e^2)+\dot{\eta}\dot{e}(40+30\eta^2+272e\eta)+\dot{e}^2(17+68\eta^2)\\ +B_1\{\ddot{D}_1D_2+2\dot{D}_1\dot{D}_2+D_1[\ddot{\eta}D_{11}+\dot{\eta}^2(72+54\eta^2)]\}\\ +B_2\{\ddot{D}_1D_3+2\dot{D}_1\dot{D}_3+D_1[\ddot{\eta}D_{12}+\dot{\eta}^2(30+30\eta^2)]\}\cos2\omega\\ +B_3\{[\dot{D}_5D_{14}+D_5(D_{18}-2D_{19})]D_4+2\dot{D}_4\dot{D}_5+D_5[\ddot{\eta}D_{13}+(45/2)\eta\dot{\eta}^2]\}\sin\omega\\ +\dot{\omega}[(7C_6+4e\dot{e}\beta^{-2})(C_3\cos\omega-2C_2\sin2\omega)+2\dot{C}_3\cos\omega-4\dot{C}_2\sin2\omega-\dot{\omega}(C_3\sin\omega+4C_2\cos2\omega)]\big)\end{aligned}$$

References

[1] Hilton, C.G. and Kuhlman, J.R., "Mathematical Models for the Space Defense Center," Philco-Ford Corporation, Colorado Springs, Colo., U-3871, Nov. 1966.

[2] Hoots, F.R. and Roehrich, R.L., "Models for Propagation of NORAD Element Sets," Aerospace Defense Command, Peterson AFB, Colo., Project SPACETRACK Rept. No. 3, Dec. 1980.

[3] Kozai, Y., "The Motion of a Close Earth Satellite," *Astronomical Journal,* Vol. 64, Nov. 1959, pp. 367-377.

[4] Lane, M.H. and Hoots, F.R., "General Perturbations Theories Derived from the 1965 Lane Drag Theory," Aerospace Defense Command, Peterson AFB, Colo., Project SPACETRACK Rept. No. 2, Dec. 1979.

[5] Lane, M.H. and Cranford, K.H., "An Improved Analytical Drag Theory for the Artificial Satellite Problem," Paper 69-925 presented at the AIAA/AAS Astrodynamics Conference, Aug. 20-22, 1969.

[6] Brouwer, D., "Solution of the Problem of Artificial Satellite Theory Without Drag," *Astronautical Journal,* Vol. 64, Nov. 1959, pp. 378-397.

[7] Hoots, F.R., "Theory of the Motion of an Artificial Earth Satellite," *Celestial Mechanics,* Vol. 23, April 1981, pp. 307-363.

[8] Lyddane, R.H., "Small Eccentricities or Inclinations in the Brouwer Theory of the Artificial Satellite," *Astronautical Journal,* Vol. 68, Oct. 1963, pp. 555-558.

[9] Vinti, J.P., "New Method of Solution for Unretarded Satellite Orbits," *Journal of Research of the National Bureau of Standards,* Vol. 62B, 1959, pp. 105-116.

[10] Hoots, F.R., "Reformulation of the Brouwer Geopotential Theory for Improved Computational Efficiency," *Celestial Mechanics,* Vol. 24, Aug. 1981, pp. 367-375.

[11] Lerch, F.J. et al., "Gravity Model Improvement Using GEOS-3 (GEM 9&10)," Goddard Space Flight Center, Greenbelt, Md., GSFC Report No. X-921-77-246, Sept. 1977.

[12] Jacchia, L.G., "New Static Models of the Thermosphere and Exosphere with Empirical Temperature Profiles," Smithsonian Astrophysical Observatory, Boston, Mass., SP-313, May 1970.